be used for determining the accuracy of approximate solutions and computational algorithms.

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SEPARATION OF THE ELASTICITY THEORY EQUATIONS WITH RADIAL INHOMOGENEITY

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The separation of a system of three elasticity theory equations in the static case to a system of two equations and one independent equation for a space with a radial inhomogeneity is presented in a spherical coordinate system. These equations are solved by separation of variables for specific kinds of radial inhomogeneity. In particular, solutions are found for the Lamé coefficients $\mu = \text{const}$, $\lambda(r)$ is an arbitrary function, $\mu = \mu_0 r^\beta$, $\lambda = \lambda_0 r^\beta$.

While methods of solving problems associated with the equilibrium of an elastic homogeneous sphere have been studied sufficiently [1], problems with spherical symmetry of the boundary conditions have mainly been solved for an inhomogeneous sphere [2, 3].

For a particular kind of inhomogeneity dependent on one Cartesian coordinate, the equations have been separated completely in [4]. A system of three equations with a radial inhomogeneity in a spherical coordinate system is separated below by a method analogous to [4].

1. The equilibrium equations in displacements with a radial inhomogeneity and no mass forces are

$$(\lambda + 2\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} - \mu\,\mathrm{rot}\,\mathrm{rot}\,\mathbf{u} + \mathbf{i}_r\lambda'\,\mathrm{div}\,\mathbf{u} + \mu\left(\mathbf{i}_r \times \mathrm{rot}\,\mathbf{u} + 2\frac{\partial u}{\partial r}\right) = 0 \qquad (1.1)$$

Here $\lambda(r)$ and $\mu(r)$ are the Lamé coefficients dependent on the radius, $\mathbf{i}_r$ is the unit vector in the radial direction, and u is the displacement vector. Let us write (1.1) in matrix form in spherical coordinates

$$\|a_{ik}\|\,\mathrm{col}\,(u_r, u_\theta, u_\varphi) = 0 \qquad (1.2)$$

$$a_{11} = \mu\,[D_\theta^\circ D_\theta + D_\varphi^2] + \frac{\partial}{\partial r}\left[\lambda D^\circ + 2\mu\frac{\partial}{\partial r}\right] + \frac{4\mu}{r}\left[\frac{\partial}{\partial r} - \frac{1}{r}\right]$$

$$a_{12} = \left[\zeta_1 \frac{\partial}{\partial r} - \frac{2\mu}{r} + \lambda'\right] D_\theta^\circ, \qquad a_{13} = \left[\zeta_1 \frac{\partial}{\partial r} - \frac{2\mu}{r} + \lambda'\right] D_\varphi$$

$$a_{22} = \zeta_1 D_\theta D_\theta^\circ + \mu\left[\Delta - \frac{1}{r^2 \sin^2\theta}\right] + \mu'\left[\frac{\partial}{\partial r} - \frac{1}{r}\right]$$

$$a_{21} = D_\theta \left[\zeta_1 D^\circ + \frac{2\mu}{r} + \mu'\right], \qquad a_{23} = D_\varphi \left[\zeta_1 D_\theta - \frac{2\mu \operatorname{ctg}\theta}{r}\right]$$

$$a_{33} = \zeta_1 D_\varphi^2 + \mu\left[\Delta - \frac{1}{r^2 \sin^2\theta}\right] + \mu'\left[\frac{\partial}{\partial r} - \frac{1}{r}\right]$$

$$a_{31} = D_\varphi \left[\zeta_1 D^\circ + \frac{2\mu}{r} + \mu'\right], \qquad a_{32} = D_\varphi \left[\zeta_1 D_\theta^\circ + \frac{2\mu}{r} \operatorname{ctg}\theta\right]$$

Here

$$\zeta_1 = \lambda + \mu, \quad \zeta = \lambda + 2\mu$$

$$D_\theta v = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad Dv = \frac{1}{r}\frac{\partial}{\partial r}(rv), \qquad D_\varphi v = \frac{1}{r\sin\theta}\frac{\partial v}{\partial \varphi}$$

$$D_\theta^\circ v = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cdot v), \qquad D^\circ v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v)$$

We introduce the following substitution (the sum of the curl and the gradient on a spherical surface):

$$u_\theta = \frac{1}{\sin\theta}\frac{\partial N}{\partial\varphi} + \frac{\partial F}{\partial\theta}, \qquad u_\varphi = -\frac{\partial N}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial F}{\partial\varphi}$$

The matrix equation consequently becomes

$$a_{11}u_r + \zeta_1 \frac{\partial}{\partial r}\left(\frac{1}{r}\Delta_* F\right) - \frac{2\mu}{r^2}\Delta_* F + \frac{\lambda'}{r}\Delta_* F = 0 \tag{1.3}$$

$$rD_\theta Q_1 + rD_\varphi Q_2 = 0, \qquad rD_\varphi Q_1 - rD_\theta Q_2 = 0 \tag{1.4}$$

$$Q_1 = \left[\frac{\zeta_1}{r}D^\circ + \frac{2\mu}{r^2} + \frac{\mu'}{r}\right]u_r + \left[\mu\Delta + \mu'\left(\frac{\partial}{\partial r} - \frac{1}{r}\right) + \frac{\zeta}{r^2}\Delta_*\right]F$$

$$Q_2 = \left[\mu\Delta + \mu'\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\right]N, \qquad \Delta_* = r^2\left[D_\theta^\circ D_\theta + D_\varphi^2\right]$$

where Δ^* is the Beltrami operator [5].

The following, easily-confirmable, relationships were used in deriving the equations:

$$\mu\left(\Delta - \frac{1}{r^2\sin^2\theta}\right) rD_\theta - 2\mu \operatorname{ctg}\theta D_\varphi^2 = rD_\theta \mu\Delta$$

$$\mu\left(\Delta - \frac{1}{r^2\sin^2\theta}\right) rD_\varphi + 2\mu \operatorname{ctg}\theta D_\varphi^2 = rD_\varphi \mu\Delta$$

One of the equations of the system (1.4) is satisfied identically if we define

$$Q_1 = -\frac{\partial w}{\partial\varphi}, \qquad Q_2 = \sin\theta \frac{\partial w}{\partial\theta} \tag{1.5}$$

Then the other equation of this system yields an equation for w

$$\Delta_* w = 0 \tag{1.6}$$

Let us now consider the solution of the system (1.3), (1.5) and (1.6) instead of the solution of the system (1.3), (1.4).

We will show that the function can be considered zero without limiting the generality. We denote by u_r°, F°, N° the solution of the homogeneous equations, and by u_r^+, F^+, N^+ the solution of the inhomogeneous equations. Let us take the following as a particular solution of the inhomogeneous equations (w_1 is the solution of (1.6)):

$$u_r^+ = 0, \qquad F^+ = -\frac{\partial}{\partial\varphi}\int_a^r w_1(\varphi, \theta, r_1) K(r_1, r)\, dr_1 \tag{1.7}$$

$$N^+ = \sin\theta \frac{\partial}{\partial\theta}\int_a^r w_1(\varphi, \theta, r_1) K(r_1, r)\, dr_1$$

$$K(r_1, r) = \frac{y_1(r_1) y_2(r) - y_2(r_1) y_1(r)}{y_1(r_1) y_2'(r_1) - y_2(r_1) y_1'(r_1)}$$

Here y_1, y_2 are linearly independent solutions of the homogeneous equation

$$\left[D^\circ \frac{\partial}{\partial r} + \mu\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\right] y = 0$$

In order to verify that F^+ and N^+ are particular solutions of the inhomogeneous equation (1.5), it is necessary to consider $\Delta_* F^+ = 0$, $\Delta_* N^+ = 0$. Therefore, the solution has the form

$$u_r = u_r^\circ, \; u_\theta = \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}(N^+ + N^\circ) + \frac{\partial}{\partial\theta}(F^+ + F^\circ) = \frac{1}{\sin\theta}\frac{\partial N^\circ}{\partial\varphi} + \frac{\partial}{\partial\theta}F^\circ$$

$$u_\varphi = -\frac{\partial}{\partial\theta}(N^+ + N^\circ) + \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}(F^+ + F^\circ) = -\frac{\partial}{\partial\theta}N^\circ + \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}F^\circ$$

The system of three equations (1.3), (1.4) has therefore been reduced to a system of two equations consisting of (1.3) and the equation $Q_1 = 0$ and an individual equation $Q_2 = 0$. In the presence of mass forces with potentials, appropriate terms describing the separation of vectors under consideration will appear in the right sides of the equations.

2. Let us consider application of the method of separation of variables to the equation $Q_2 = 0$ describing the shear strain. It is natural to seek the solution as

$$N = \Sigma f_n(r) Y_n(\theta, \varphi)$$

where $(Y(\theta, \varphi)$ is a spherical function. Using a property of spherical functions [1], we obtain a second order equation for $f_n(r)$

$$f_n''(r) + f_n'(r)\left(\frac{\mu'}{\mu} + \frac{2}{r}\right) - f_n(r)\left[\frac{(n+1)n}{r^2} + \frac{\mu'}{\mu r}\right] = 0 \tag{2.1}$$

Of special interest are solutions of (2.1) which are expressed in terms of known functions. For this we substitute

$$f_n(r) = f_* / (r\sqrt{\mu})$$

and reduce (2.1) to normal form [6]

$$f_*'' + \left\{\frac{-(n+1)n}{r^2} - \frac{\mu''}{2\mu} + \frac{1}{4}\left(\frac{\mu'}{\mu}\right)^2 - \frac{2\mu'}{\mu r}\right\} f_* = 0 \tag{2.2}$$

According to [6], the solution of (2.2) is related to the solution of the equation

$$\frac{d^2}{dx^2}\eta + P(x)\eta = 0$$

where $\eta = \eta(x)$, the dependence $f_*(r) = \eta(x) / \sqrt{x'_r}$, where $x = x(r)$ if the following relationship is satisfied

$$\frac{1}{2}D_1\{x_r', r\} + (x_r')^2 P(x) = -\frac{(n+1)\, n}{r^2} - \frac{\mu''}{2\mu} + \frac{1}{4}\left(\frac{\mu'}{\mu}\right)^2 - \frac{2\mu'}{\mu r} = \tag{2.3}$$

$$-\frac{(n+1)n - 2}{r^2} + \frac{1}{2}D_1\left\{\frac{1}{\mu r^4}, r\right\}, \qquad D_1\{x_r', r\} = -2(x_r')^{1/2}\frac{d^2}{dr^2}(x_r')^{-1/2}$$

here $(D_1\{x_r', r\}$ is the Schwartz derivative [6]. Using the easily verifiable property of the Schwartz derivative

$$D_1\{W_r', r\} = D_1\{x_r', r\} + D_1\{W_x', x\}(x_r')^2 \qquad (2.4)$$

where $W = W[x(r)]$, the following identity can be written

$$D_1\left\{\frac{1}{\mu r^4}, r\right\} = D_1\{x_r', r\} + D_1\left\{\frac{r_x'}{\mu r^4}, x\right\}(x_r')^2 \qquad (2.5)$$

(In the notation under consideration $W_r' = 1/(\mu r^4)$.) Taking account of the identity (2.5), we reduce the relationship (2.3) to two equations

$$-\frac{(n+1)n-2}{r^2}(r_x')^2 = P_1(x), \qquad \frac{1}{2}D_1\left\{\frac{r_x'}{r^4\mu}, x\right\} = P_2(x) \qquad (2.6)$$

$$(P_1(x) + P_2(x) = P(x))$$

We write the last equation in (2.6) explicitly by using the following definition of the Schwartz derivative

$$\left[\frac{d^2}{dx^2} + P_2(x)\right]\left[\frac{r^4\mu}{r_x'}\right]^{1/2} = 0 \qquad (2.7)$$

Using the normal form of the Helmholtz, Bessel and Whittaker equations, i.e. giving the function $P(x)$ and partitioning it into the parts $P_1(x)$ and $P_2(x)$ convenient for calculations, we find the function $\mu(r)$ from the first equation of (2.6) and (2.7), for which the solutions of (2.2) are expressed in terms of the solutions of the corresponding equation.

Let us examine specific examples.

(1). The solutions are expressed in terms of the Helmholtz equations. In this case $P(x) = -s^2$. We hence assume $P_1(x) = -s_1^2$, $P_2(x) = -s_2^2$ so that $-s_1^2 - s_2^2 = -s^2$. From (2.6) we obtain two relations

$$-(n+1)n + 2 = -s_1^2, \qquad r_x' = r$$

The last relation determines r as a function of x, $r = ce^x$, where c is the constant of integration. Substituting $P_2(x) = -s_2^2$ into (2.7) and solving it with respect to $r^4\mu/r_x'$, we obtain

$$\mu = (A_1e^{-s_2x} + A_2e^{s_2x})^2/r^3$$

or respectively

$$\mu = (A_1r^{-s_2} + A_2r^{s_2})^2/r^3$$

We find for the function f_*

$$f_* = \sqrt{r}\,(B_1r^{-\sqrt{(n+1)n-2+s_2^2}} + B_2r^{\sqrt{(n+1)n-2+s_2^2}})$$

In this case the function $f_n(r)$ is

$$f_n(r) = \frac{r}{(A_1r^{-s_2} + A_2r^{s_2})}(B_1r^{-\sqrt{(n+1)n-2+s_2^2}} + B_2r^{\sqrt{(n+1)n-2+s_2^2}})$$

For $A_1 = 0$, $s_2 = 3/2$, the solutions go over into known solutions corresponding to a homogeneous space.

(2). Solutions expressed in terms of solutions of the Bessel equation. In this case, we obtain from the first equation of (2.6) and (2.7) for $P_1(x) = (1-4\nu^2)/(4x^2)$, $P_2(x) = s^2$, where ν is the order of the Bessel function

$$r = x, \qquad \nu = \pm\sqrt{n^2 + n - 7/4}, \qquad \mu = \{A_1e^{-isr} + A_2e^{isr}\}^2/r^4$$

For an appropriate $P(x)$ we find for the function f_* from (2.3)

$$f_*(r) = [B_1 J_\nu(sr) + B_2 J_{-\nu}(sr)]\sqrt{r}$$

If we assume

$$P_1(x) = s^2, \qquad P_2(x) = \frac{1-4\nu^2}{4x^2}$$

we obtain from the first equation of (2.6) and (2.7) ($s^2 = 2 - (n+1)n$ and $\ln r = x$)

$$(\mu r^3)^{1/2} = \begin{cases} c_1 x^{1/2+k} + c_2 x^{1/2-k} \; (k = |\nu|), & \nu^2 > 0 \\ c_1\sqrt{x} + c_2\sqrt{x}\ln x, & \nu^2 = 0 \\ c_1\sqrt{x}\cos(k\ln x) + c_2\sqrt{x}\sin(k\ln x), & \nu^2 < 0 \end{cases}$$

The functions $f_*(r)$ and $f_n(r)$ are expressed in terms of Bessel functions in conformity with the examples presented.

An analysis of solutions expressed in terms of the hypergeometric function is carried out analogously.

3. To solve the system consisting of the two equations: $Q_1 = 0$ and (1.3), let us use separation of variables. Let us seek the solution in the form $u_r = \Sigma u_n Y_n(Q, \varphi)$, $F = \Sigma F_n Y_n(\theta, \varphi)$. We consequently obtain a system of ordinary differential equations (3.1) which we write in matrix form

$$\left[E\frac{d^2}{dr^2} - G\frac{d}{dr} - H\right]\begin{Vmatrix} u_n \\ F_n \end{Vmatrix} = 0 \qquad (3.1)$$

$$G = \begin{Vmatrix} -\left(\frac{2}{r} + \frac{\zeta'}{\zeta}\right) & \frac{\zeta_1(n+1)n}{\zeta r} \\ \frac{\zeta_1}{\mu r} & -\left(\frac{2}{r} + \frac{\mu'}{\mu}\right) \end{Vmatrix}$$

$$H = \begin{Vmatrix} \frac{2}{r^2} + \frac{\Gamma(n+1)n\mu}{r^2\zeta} - \frac{2\lambda'}{r} & -\frac{(n+1)n(\zeta+\mu)}{r^2\zeta} + \frac{(n+1)n\lambda'}{\zeta r} \\ -\left(\frac{2\zeta}{\mu r^2} + \frac{\mu'}{\mu r}\right) & \frac{\zeta(n+1)n}{r^2\mu} + \frac{\mu'}{\mu r} \end{Vmatrix}$$

where E is the unit matrix.

Let us examine the simplest cases of solving the system (3.1).

(1). Inhomogeneous space with $\lambda = \lambda_0 r^\beta$, $\mu = \mu_0 r^\beta$. Substituting λ and μ in the matrices G and H, we obtain that the matrix of the coefficients is inversely proportional to r, and $H \sim 1/r^2$. Consequently, the system is a matrix equation of Euler type, and its solutions are sought as $u_n = u_* r^m$, $F_n = F_* r^m$, where u_* and F_* are constants, and m are the roots of the fourth order characteristic equation obtained as a result of substituting the solutions into (3.1).

In particular, for $\beta = 0$ we obtain known values of m corresponding to solutions for a homogeneous space.

(2). Let us consider the case when one of the solutions of (1.1) is $u = \operatorname{grad}\chi$. In the notation of the system (3.1), we obtain $u_n = \partial\chi/\partial r$, $F_n = \chi/r$. Substituting these expressions into (3.1), we obtain two equations in χ; the condition for identity of these equations is governed by the law of inhomogeneity of the media for which one of the solutions has the form mentioned. The identity condition is (two cases)

a) when

$$\zeta = \frac{2(\mu')^2 r}{\mu'' r - \mu'} \qquad (3.2)$$

b) when $\mu = \text{const}$ and λ is an arbitrary function. When $\mu = \text{const}$, we represent (3.1) with respect to the variables u_n and $\chi = F_n r$ as

$$\left[\frac{d}{dr} - A\right]\left[\frac{d}{dr} - B\right] \left\|\begin{matrix} u_n \\ \chi \end{matrix}\right\| = 0 \qquad (3.3)$$

A solution of (3.3) is $\Psi_b, \Psi_b \int \Psi_b^{-1}\Psi_a dr$, where Ψ_b and Ψ_a are matrices of the fundamental solutions of the equations

$$\left[\frac{d}{dr} - A\right]\Psi_a = 0, \qquad \left[\frac{d}{dr} - B\right]\Psi_b = 0 \qquad (3.4)$$

$$A = \left\|\begin{matrix} -\frac{\lambda'}{\zeta} & -\frac{\mu(n+1)n}{\zeta r^2} \\ -\frac{\zeta}{\mu} & 0 \end{matrix}\right\|, \quad B = \left\|\begin{matrix} -\frac{2}{r} & \frac{(n+1)n}{r^2} \\ 1 & 0 \end{matrix}\right\|$$

Solutions of the system (3.4) are

$$\Psi_a = \left\|\begin{matrix} \frac{-\zeta(n+1)nr^n}{\mu} & \frac{\zeta n r^{-(n+1)}}{\mu} \\ r^{n+1} & r^{-n} \end{matrix}\right\|, \quad \Psi_b = \left\|\begin{matrix} nr^{n-1} & -(n+1)r^{-(n+2)} \\ r^n & r^{-(n+1)} \end{matrix}\right\|$$

When the relationship (3.2) is satisfied, the representation of (3.1) in the form (3.3) yields

$$A = \left\|\begin{matrix} \frac{-\lambda'}{\zeta} & \frac{-\mu(n+1)n}{\zeta r^2} - \frac{2\mu'}{\zeta r} \\ \frac{-\zeta'}{\mu} & \frac{-\mu'}{\mu} \end{matrix}\right\|, \quad B = \left\|\begin{matrix} -\left(\frac{1}{r} + \frac{\mu''}{\mu}\right) & \frac{(n+1)n-1}{r^2} + \frac{\mu''}{\mu' r} \\ 1 & 0 \end{matrix}\right\|$$

The solutions of equations (3.4) for the matrices A and B under the condition (3.2) have not been given successfully in closed form, as it was done for $\mu = \text{const}$. Only in particular cases can the solutions be expressed in terms of known solutions of second-order equations.

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